

ON THE PRIME COUNTING FUNCTION AND THE PARTIAL SUM OF RECIPROCAL OF ODD PRIMES

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ABSTRACT. We present a function that tests for primality, factorizes composites and builds a closed form expression of $\pi(n^2)$ in terms of $\sum_{3 \leq p \leq n} \frac{1}{p}$ and a weaker version of $\omega(n)$.

KEYWORDS:

Primality test; Integer factorization; Prime counting function; Prime factors; Partial sum of reciprocals of primes.

1. INTRODUCTION

The prime counting function $\pi(n)$, the partial sum of the reciprocals of odd primes $\sum_{3 \leq p \leq n} \frac{1}{p}$, and $\omega(n)$ the number of distinct prime factors of n , have historic interests for being well studied by Euler[3], Hardy, Ramanujan[7], and Erdos[5]. We establish an equation that unites these three functions.

Consider a function $f(n, x)$ that takes two positive odd integers n and x and returns the smallest odd multiple of x exceeding n . Example: $f(1, 3) = 3$, $f(5, 3) = 9$. We observe that, the minimum of $f(n, p)$ taken over odd primes $p \leq \lceil \sqrt{n} \rceil$, is the smallest odd composite exceeding n ; call this c_1 .

If n is odd, then $n + 2$ is a prime if and only if the gap between c_1 and n is 4 or 6. Hence, one has a primality test based on computing $f(n, x)$. We also point out that, if n and x are positive odd integers, then $x | n$ if and only if $f(n - 2, x) = n$; this gives a second primality test and factorization criterion.

Furthermore, $f(n, x)$ is used to count the number of odd integers in the interval $(n^2, (n + 2)^2]$. This count is set to be equal to the expected number of odd integers in the same interval given by $\frac{(n+2)^2 - n^2}{2}$. The resulting equality gives an exact formula of $\sum_{3 \leq p \leq n} \frac{1}{p}$ and the outcome in this argument is illustrated by examples and applications.

We exhibit a closed form expression of $\pi(n^2)$ in terms of $\sum_{3 \leq p \leq n} \frac{1}{p}$, and a weaker version of $\omega(n)$, and compare the expression of $\pi(n)$ provided by the prime number Theorem with $\pi(n^2)$ that rises from counting the odd integers in the interval $(1, n^2]$.

2. INTRODUCING THE FUNCTION $f(n, x)$

Definition 2.1.

Given a positive odd integer $n > 3$, let A_n denote the set of all odd primes less than or equal to the ceiling of square root of n . That is, $A_n = \{3 \dots \lceil \sqrt{n} \rceil\}$. Define a function

$$f(n, x) = n + 2x - (n - x) \bmod(2x), \text{ where } x \in A_n. \quad (2.1)$$

For each prime x in A_n , the function $f(n, x)$ yields the smallest odd integer multiple of x greater than n .

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Example 2.2. let $n = 81$, we have $\lceil \sqrt{n} \rceil = 9$. Hence, $A_n = \{3.5.7\}$.

$$f(81, 3) = 81 + 2(3) - [(81 - 3) \bmod (2 * 3)] = 87.$$

$$f(81, 5) = 81 + 2(5) - [(81 - 5) \bmod (2 * 5)] = 85.$$

$$f(81, 7) = 81 + 2(7) - [(81 - 7) \bmod (2 * 7)] = 91.$$

$f(81, 3) = 87$, is the smallest odd multiple of 3 greater than 81.

$f(81, 5) = 85$, is the smallest odd multiple of 5 greater than 81.

$f(81, 7) = 91$, is the smallest odd multiple of 7 greater than 81.

Example 2.3. For $n = 111$, the set of all odd primes less than or equal to $\lceil \sqrt{111} \rceil$ is $A_n = \{3, 5, 7, 11\}$. Note: The prime number 11 is in the set A_n because of the ceiling of square root of n .

$f(n, x) = n + 2x - [(n - x) \bmod (2x)]$ where x is in A_n .

$$f(111, 3) = 111 + 2(3) - [(111 - 3) \bmod (2 * 3)] = 117.$$

$$f(111, 5) = 111 + 2(5) - [(111 - 5) \bmod (2 * 5)] = 115.$$

$$f(111, 7) = 111 + 2(7) - [(111 - 7) \bmod (2 * 7)] = 119.$$

$$f(111, 11) = 111 + 2(11) - [(111 - 11) \bmod (2 * 11)] = 121.$$

$f(111, 3) = 117$, is the smallest odd multiple of 3 greater than 111.

$f(111, 5) = 115$, is the smallest odd multiple of 5 greater than 111.

$f(111, 7) = 119$, is the smallest odd multiple of 7 greater than 111.

What the function $f(n, x)$ does? The value $(n - x) \bmod (2x)$ is the distance between n , to the largest odd multiple of x , less than or equal to n ; therefore, $n - [(n - x) \bmod (2x)]$ is the largest odd multiple of x less than or equal to n . Add the distance $2x$ to the largest odd multiple of x less than or equal to n , this gives the smallest odd multiple of x that is greater than n .

Lemma 2.4. Given two positive odd integers n and x , the function $f(n, x)$ yields the smallest odd multiple of x exceeding n .

Proof. (By contradiction), choose two positive odd integers n and x , and suppose that $f(n, x)$ is not the smallest odd multiple of x that is greater than n . This means that there exists an integer k_2 such that $n < k_2x < f(n, x)$, that is,

$$(2) \quad k_2x < n + 2x - [(n - x) \bmod (2x)].$$

But since $n - [(n - x) \bmod (2x)]$ is the largest odd multiple of x less than or equal to n , there exists an integer k_1 such that $n - [(n - x) \bmod (2x)] = k_1x$. Thus, inequality (2) becomes $k_2x < 2x + k_1x$, which implies that $k_2 - k_1 < 2$. This is impossible because $(k_2x$ is an odd multiple of x that is) $> n$, and $(k_1x$ is an odd multiple of x that is) $\leq n$, so the value $k_2 - k_1$ cannot be less than 2. Therefore, given two positive odd integers n and x , the function $f(n, x)$ gives rise to the smallest odd multiple of x exceeding n . \square

Note: Lemma 2.4 is proven for any two arbitrary positive odd integers n and x , but we will often let x to be a prime in the interval $[3, \lceil \sqrt{n} \rceil]$.

2.1. First primality test algorithm.

Given a positive odd integer $n > 3$, Let p_n denote the largest prime less than or equal to $\lfloor \sqrt{n} \rfloor$. We build a list L that contains the elements of $f(n, p)$ where p is a prime limited to $3 \leq p \leq p_n$. Thus, $L = \{f(n, 3), f(n, 5), \dots, f(n, p_n)\}$, in other words, the elements of L are, {the smallest odd multiple of 3 exceeding n , the smallest odd multiple of 5 exceeding n , ..., the smallest odd multiple of p_n exceeding n } not necessarily in this particular order. When L is sorted in an increasing order, the first element in the sorted list is the minimum of the odd composites exceeding n . Let c_1 denote this composite, **we say that c_1 is the smallest odd composite exceeding n** . This means that if there is any odd integer k between n and c_1 , then k cannot be a composite. Therefore, k must be a prime.

In example 1 page 3, the sorted list $L = \{85, 87, 91\}$, this means $c_1 = 85$ is the smallest odd composite that is greater than $n = 81$. Now since there is a gap between n and c_1 wide enough to hold one odd integer, we say that the odd integer in this gap that is $n + 2 = 81 + 2 = 83$ cannot be a composite; hence, it is a prime.

Theorem 2.5. *If n is a positive odd integer, then $n + 2$ is a prime if and only if the gap between c_1 and n is 4 or 6.*

Proof. Choose a positive odd integer n , and suppose that $n + 2$ is a prime. The integer c_1 is the smallest odd composite greater than n . This means that $n - c_1$ must be equal to at least 4; thus, $n - c_1 = 4$ or 6. (This gap between n and c_1 cannot be more than 6 as we explained in a Note right below this proof on page 3).

Conversely, assume that $c_1 - n = 4$ or 6. Since c_1 is the smallest odd composite greater than n , we can state the following. If $c_1 - n = 4$, then the positive odd integer $n + 2$ that is between n and c_1 is not a composite; thus, $n + 2$ is a prime. Similarly, if $c_1 - n = 6$, then the two positive odd integers $n + 2$ and $n + 4$ that are between n and c_1 are not composites. Hence, $n + 2$ and $n + 4$ must be a pair of twin primes. This means that in whichever circumstance the case applies, we have $n + 2$ is a prime. We conclude that $n + 2$ is a prime if and only if the gap between c_1 and n is either 4 or 6. \square

Note: The gap between n and c_1 cannot be greater than 6 since it is well known that one number must be a multiple of 3 in every sequence of three consecutive odd numbers.

Corollary 2.6. *Given a positive odd integer n , we can prove that $n + 2$ is a lower member of a pair of twin primes if and only if the gap between n and c_1 is 6.*

Proof. Assume that the gap between n and c_1 is 6, this implies that there are two positive odd integers between n and c_1 that are not composite numbers. Hence, they are a pair of twin primes and $n + 2$ is the lower member of the pair.

Conversely, suppose that $n + 2$ is a lower member of a pair of twin primes, this means that $n + 4$ is also a prime. Thus, the gap between n and c_1 is 6. We conclude that given a positive odd integer n , we have $n + 2$ is a lower member of a pair of twin primes if and only if the gap between n and c_1 is 6. \square

Corollary 2.7. *Given a positive odd integer n , we can prove that $n + 2$ is a composite if and only if the gap between n and c_1 is 2.*

Proof. Assume that $n + 2$ is a composite, so $n + 2$ is the smallest odd composite that is greater than n , but this title belongs to c_1 . Hence, $n + 2 = c_1$. This means that the gap between n and c_1 is 2. Conversely, suppose that the gap between n and c_1 is 2. This means that $n + 2 = c_1$

which is a composite. Therefore, $n + 2$ is a composite if and only if the gap between n and c_1 is 2. \square

Example 2.8. For $n = 111$, the set of all odd primes less than or equal to $\lceil \sqrt{111} \rceil$ is $A_n = \{3, 5, 7, 11\}$. $f(n, x) = n + 2x - [(n - x) \bmod (2x)]$ where x is in A_n .

$$f(111, 3) = 111 + 2(3) - [(111 - 3) \bmod (2 * 3)] = 117.$$

$$f(111, 5) = 111 + 2(5) - [(111 - 5) \bmod (2 * 5)] = 115.$$

$$f(111, 7) = 111 + 2(7) - [(111 - 7) \bmod (2 * 7)] = 119.$$

$$f(111, 11) = 111 + 2(11) - [(111 - 11) \bmod (2 * 11)] = 121.$$

The sorted list $L = \{115, 117, 119, 121\}$, so $c_1 = 115$. This means that $c_1 - n = 115 - 111 = 4$; thus, $n + 2 = 113$ is a prime by Theorem 2.5.

Example 2.9. For $n = 189$, the set of all odd primes less than or equal to $\lceil \sqrt{189} \rceil$ is $A_n = \{3, 5, 7, 11, 13\}$. $f(n, x) = n + 2x - [(n - x) \bmod (2x)]$ where x is in A_n .

$$f(189, 3) = 189 + 2(3) - [(189 - 3) \bmod (2 * 3)] = 195.$$

$$f(189, 5) = 189 + 2(5) - [(189 - 5) \bmod (2 * 5)] = 195.$$

$$f(189, 7) = 189 + 2(7) - [(189 - 7) \bmod (2 * 7)] = 203.$$

$$f(189, 11) = 189 + 2(11) - [(189 - 11) \bmod (2 * 11)] = 209.$$

$$f(189, 13) = 189 + 2(13) - [(189 - 13) \bmod (2 * 13)] = 195.$$

The sorted list $L = \{195, 195, 195, 203, 209\}$, so that $c_1 = 195$. Thus, $c_1 - n = 195 - 189 = 6$; therefore, $n + 2 = 191$ is a lower member of a pair of twin primes by Corollary 2.6.

Example 2.10. For $n = 297$, the set of all odd primes less than or equal to $\lceil \sqrt{297} \rceil$ is $A_n = \{3, 5, 7, 11, 13, 17\}$. $f(n, x) = n + 2x - [(n - x) \bmod (2x)]$ where x is in A_n .

$$f(297, 3) = 297 + 2(3) - [(297 - 3) \bmod (2 * 3)] = 303.$$

$$f(297, 5) = 297 + 2(5) - [(297 - 5) \bmod (2 * 5)] = 305.$$

$$f(297, 7) = 297 + 2(7) - [(297 - 7) \bmod (2 * 7)] = 301.$$

$$f(297, 11) = 297 + 2(11) - [(297 - 11) \bmod (2 * 11)] = 319.$$

$$f(297, 13) = 297 + 2(13) - [(297 - 13) \bmod (2 * 13)] = 299.$$

$$f(297, 17) = 297 + 2(17) - [(297 - 17) \bmod (2 * 17)] = 323.$$

The sorted list $L = \{299, 301, 303, 305, 319, 323\}$, so $c_1 = 299$. Hence, $c_1 - n = 299 - 297 = 2$; therefore, $n + 2 = 299$ is a composite by Corollary 2.7.

3. PRIMALITY TESTING AND INTEGER FACTORIZATION USING $f(n, x)$ AS A STANDALONE FUNCTION

3.1. $f(n, x)$ is an integer factorization function.

Theorem 3.1. *Given two positive odd integers n and x , we can prove that x is a factor of n if and only if x satisfies the equation $f(n - 2, x) = n$.*

Proof. Let n and x be two positive odd integers and suppose that x is a factor of n ; thus, n is the smallest odd multiple of x that is greater than $n - 2$. By Lemma 2.4 on page 2, this means that $f(n - 2, x) = n$.

Conversely, assume that $f(n - 2, x) = n$. By Lemma 2.4, $f(n - 2, x)$ gives the smallest odd multiple of x greater than $n - 2$. This means that $f(n - 2, x) = kx$ for some integer k . From our hypothesis we have $f(n - 2, x) = n$, and now we have $f(n - 2, x) = kx$; these two equations imply that $kx = n$, in other words x is a factor of n . Therefore, given two positive odd integers n and x , it holds true that x is a factor of n if and only if x satisfies the equation $f(n - 2, x) = n$. \square

Example 3.2. *Using "Walfram, mathematica"*

*What are the prime factors of $n = 15015 = 3 * 5 * 7 * 11 * 13$?*

Solution: *Given $n = 15015$, Theorem 3.1 in subsection 3.1 states that the integer solutions of the equation $f(n - 2, x) = n$ are the factors of n . But since we know that the factors of n are less than or equal to n therefore, we can look for x in the interval between 1 and n this is $1 \leq x \leq n$. Enter the following equation:*

$$15013 + 2x - [(15013 - x) \bmod (2x)] = 15015, \quad 1 \leq x \leq 15015, \quad x \text{ is an integer.}$$

The solutions will be $x = 1, x = 3, x = 5, x = 7, x = 11, x = 13, x = 15, x = 21$. See Table 1 for the list of solutions using Excel.

TABLE 1. *Example 5 solutions using Excel: The primes x_i where $f(n - 2, x_i) - n = 0$ are prime factors of n .*

n	x_i	$f(n - 2, x) - n$
15015	1	0
	3	0
	5	0
	7	0
	9	6
	11	0
	13	0
	15	0
	17	30
	19	14
	21	0
	23	4

	15015	0

Example 3.3. *Find the prime factors of $n = 7663 = 79 * 97$.*

Solution: *Given $n = 7663$, the integer solutions of $f(n - 2, x) = n$ are all factors of n . Enter the following equation:*

$$7661 + 2x - [(7661 - x) \bmod (2x)] - 7663, 1 \leq x \leq 7663, x \text{ is an integer.}$$

The solutions will be $x = 1, x = 79, x = 97, x = 7663$. See also Table 2 on page 7 for all solutions using Excel.

TABLE 2. Example 6 solutions using Excel: The primes x_i where $f(n - 2, x_i) - n = 0$ are prime factors of n .

n	x_i	$f(n - 2, x) - n$
7663	1	0
	3	2
	5	2
	7	2
	9	14

	77	144
	79	0
	81	32

	95	32
	97	0
	99	158

	7663	0

3.2. $f(n, x)$ is an unconditional general-purpose deterministic primality test.

Theorem 3.4. A positive odd integer n greater than 1 is a prime if and only if the equation $f(n - 2, x) = n$ has only two solutions 1 and n .

Proof. Suppose that a positive odd integer n is a prime, by virtue of the definition of a prime number, n has only two factors 1 and n . But all factors of n are the x solutions of the equation $f(n - 2, x) = n$ as a result of the factorization Theorem in subsection 3.1. Thus, the equation $f(n - 2, x) = n$ has only two solutions 1, and n .

Conversely, suppose that the equation $f(n - 2, x) = n$ has only two solutions 1 and n . By the factorization Theorem in subsection 3.1 on page 6, these solutions 1 and n are all the factors of n . Hence, n has only two factors 1 and n , this means that n is a prime as a result of the definition of a prime number. We conclude that an odd integer n greater than 1 is a prime if and only if the equation $f(n - 2, x) = n$ has only two solutions 1 and n . \square

Example 3.5. Using "Walfram, mathematica", is $n = 139$ a prime?

Solution: Given $n = 139$, the integer solutions of $f(n - 2, x) = n$ are all the positive integer factors of n by the factorization Theorem in subsection 3.1. Enter the following equation

$$137 + 2x - [(137 - x) \bmod (2x)] = 139, 1 \leq x \leq 139, x \text{ is an integer.}$$

The solutions will be $x = 1, x = 139$. This means that 139 has no factor other than 1 and itself. Hence, 139 is a prime.

Example 3.6. is $n = 3913 = 7 * 13 * 43$ a prime?

Solution: the integer solutions of $f(n - 2, x) = n$ are all the positive integer factors of n by the factorization Theorem in subsection 3.1. Enter the following equation,

$$3911 + 2x - [(3911 - x) \bmod (2x)] = 3913, \quad 1 \leq x \leq 3913, \quad x \text{ is an integer.}$$

The solutions are $x = 1, x = 7, x = 13, x = 43, x = 91, x = 301, x = 559, x = 3913$.

Thus, 3913 is not a prime. Another way to have all the solutions is to make a table, choose x such that $1 \leq x \leq n$ and collect the values of x for which the equation $f(n - 2, x) = n$ holds as we did in Examples 5 and 6 using Excel.

3.3. Discussion on runtime analysis. The function $f(n, x)$ has a runtime that scales poorly compared to some other primality test algorithms such as the AKS algorithm [1], but it has some attractions and similarities with the trial division.

- The function $f(n, x)$ is an unconditional general-purpose deterministic primality test.
- The function $f(n, x)$ is a standalone algebraic function not a step by step finite sequence of logical instructions like an algorithm.
- The function $f(n, x)$ performs both, primality test and integer factorization. (It is similar to the trial division in that sense.)
- The function $f(n, x)$ gives the exact value of $\omega(n)$ the number of distinct prime factors of n (similar to the trial division). Indeed the $\omega(n)$ is equal to the number of prime solutions of the equation $f(n - 2, x) = n$ as proven in subsection 3.1 and illustrated by examples 5 and 6.
- The implementation of $f(n, x)$ in a computer is very simple. "If there was an algebraic function equivalent to the trial division, then $f(n, x)$ would be very similar to that function."

- A major advantage of $f(n, x)$ over the trial division is that $f(n, x)$ is a closed form expression and it can be used as a counting function. In the following process, we shall demonstrate how $f(n, x)$ is used to count the number of odd integers in a given interval. This counting process facilitates a rise of a relation between $\pi(n^2)$, $\sum_{3 \leq p \leq n} \frac{1}{p}$, and a weaker version of $\omega(n)$.

4. SOME APPLICATIONS OF $f(n, x)$ AS A COUNTING FUNCTION

4.1. Counting the odd integers in the interval $(n^2, (n + 2)^2]$. This leads to an exact formula of $\sum_{3 \leq x_i \leq n} \frac{1}{x_i}$. We count the number of odd composites in the interval $(n^2, (n + 2)^2]$, and we add the number of primes that are in the same interval to obtain the total number of odd integers in the interval $(n^2, (n + 2)^2]$. The notation $\pi(\mathcal{L})$ denotes the "Lengendre's pie": $\pi(\mathcal{L}) = \pi((n + 2)^2) - \pi(n^2)$.

Let x_n be the largest prime less than or equal to the ceiling of square root of n . The number of odd composites in the interval $(n^2, (n + 2)^2]$ is the number of odd composite multiples of 3 plus the number of odd composite multiples of 5, ..., plus the number of odd composite multiples of x_n , and we subtract the duplicates from the sum. Here is an example of duplicate counts: Say an odd integer $n_k = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ has three distinct prime factors $3 \leq p_1, p_2, p_3 \leq n$, so n_k is counted three times. First, n_k was counted as a multiple of p_1 , then as a multiple of p_2 and finally as a multiple of p_3 ; thus, it is necessary to subtract the two extra times that the integer n_k was counted.

How to count the odd composite multiples of 3 that are in the interval $(n^2, (n + 2)^2]$? These numbers start from the smallest odd composite multiple of 3 greater than n^2 to the largest odd multiple of 3 less than or equal to $(n + 2)^2$. The function $f(n, x)$ can compute the smallest odd composite multiple of 3 greater than n^2 that is $f(n^2, 3)$. It is also known that (the largest odd composite multiple of 3 less than or equal to $(n + 2)^2$) is equal to (the smallest odd composite multiple of 3 that is greater than $(n + 2)^2 - 6$ this is $f((n + 2)^2, 3) - 6$. So between the first

odd composite multiple of 3 greater than n^2 , and the last odd composite multiple of 3 less than or equal to $(n+2)^2$, every time we move 6 units, there is an odd multiple of 3. In conclusion, the number of odd composite multiples of 3 in the interval $(n^2, (n+2)^2]$ is,

$$\frac{[f((n+2)^2, 3) - 6] - [f(n^2, 3)]}{6} + 1 = \frac{f((n+2)^2, 3) - f(n^2, 3)}{6}.$$

Similarly, the number of odd composite multiples of 5 in the interval $(n^2, (n+2)^2]$ is,

$$\frac{[f((n+2)^2, 5) - 10] - [f(n^2, 5)]}{10} + 1 = \frac{f((n+2)^2, 5) - f(n^2, 5)}{10}.$$

Hence, in general, given a prime x_i such that $3 \leq x_i \leq n$, the number of odd composite multiples of x_i that are in the interval $(n^2, (n+2)^2]$ is,

$$\frac{f((n+2)^2, x_i) - f(n^2, x_i)}{2x_i}.$$

Thus, the number of odd composites in the interval $(n^2, (n+2)^2]$ is,

$$\left(\sum_{3 \leq x_i \leq n} \frac{f((n+2)^2, x_i) - f(n^2, x_i)}{2x_i} \right) - \text{dup}.$$

$$\text{dup} = \sum_{n^2 < n_k \leq (n+2)^2} \text{dup}(n_k).$$

$$\text{dup}(n_k)_{n^2 < n_k \leq (n+2)^2} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases}$$

where m is the number of distinct prime factors less than or equal to n , of the odd integer n_k . Note: m is simply a weaker version of $\omega(n_k)$.

Let's give a more comprehensive explanation of the dup function. Suppose that n_k is an odd integer in the interval $(n^2, (n+2)^2]$ and $n_k = p_1^{s_1} p_2^{s_2} \dots p_r^{s_r}$ and n_k has m distinct prime factors less than or equal to n , say these prime factors are p_1, p_2, \dots, p_m , then n_k was counted as a multiple of p_1 , a multiple of p_2 , ..., and as a multiple of p_m . Thus, n_k was duplicated $m - 1$ times. Since $f(n, x)$ does not filter duplicate items, it is necessary to subtract the extra number of times n_k is counted. The sum of these extra number of times each n_k was counted, is the dup function.

The total number of odd integers in the interval $(n^2, (n+2)^2]$ is,

$$\left(\sum_{3 \leq x_i \leq n} \frac{f((n+2)^2, x_i) - f(n^2, x_i)}{2x_i} \right) - \text{dup} + \pi(\mathcal{L}).$$

Where $\pi(\mathcal{L})$ is the "Legendre's Pie". i.e. $\pi(\mathcal{L}) = \pi((n+2)^2) - \pi(n^2)$. The total number of odd integers in the interval $(n^2, (n+2)^2]$ is also,

$$\frac{(n+2)^2 - n^2}{2}.$$

Note: The function $f(n, x)$ does not always include $(n+2)^2$ in the count of the number of composites in the interval $(n^2, (n+2)^2]$. We shall explain in detail in the following two cases

why the inclusion of $(n+2)^2$ in the count depends on the primality of $n+2$.

• Case 1: If $n+2$ is a composite,
then $f(n, x)$ **will count** $(n+2)^2$ as a composite in the interval $(n^2, (n+2)^2]$. This is because $(n+2)$ is a composite, so $n+2$ is a multiple of some prime $p \leq n$, and $(n+2)^2$ is also a multiple of p . Now since x_i is a prime such that $3 \leq x_i \leq n$, this means that x_i will take the value of p at some point. Moreover, because $f(n, x)$ counts all the multiple of x_i that are in the interval $(n^2, (n+2)^2]$; it follows that $(n+2)^2$ will be counted as a multiple of $x_i = p$. Thus, if $(n+2)$ is a composite, then we can say that the number of odd integers in the interval $(n^2, (n+2)^2]$ that $f(n, x)$ has counted is equal to the expected number of odd integers in the interval $(n^2, (n+2)^2]$. This expected number is $\frac{(n+2)^2 - n^2}{2}$. The comparison translates to,

$$\left(\sum_{3 \leq x_i \leq n} \frac{f((n+2)^2, x_i) - f(n^2, x_i)}{2x_i} \right) - dup + \pi(\mathcal{L}) = \frac{(n+2)^2 - n^2}{2}. \quad (4.1)$$

• Case 2: If $n+2$ is a prime,
then $f(n, x)$ **will not count** $(n+2)^2$ as a composite in the interval $(n^2, (n+2)^2]$. This is because the prime x_i is such that $3 \leq x_i \leq n$. This means that x_i cannot be equal to $n+2$ which is the only prime factor of $(n+2)^2$. Hence, $(n+2)^2$ will not be counted as a multiple of any prime $x_i \leq n$. Therefore, we must **Add 1** to the number of odd integers counted by $f(n, x)$ to compensate the missing count of $(n+2)^2$. So if $n+2$ is a prime, then the number of odd integers in the interval $(n^2, (n+2)^2]$ that $f(n, x)$ has counted **+1**, is equal to the expected number of odd integers in the interval $(n^2, (n+2)^2]$, which is equal to, $\frac{(n+2)^2 - n^2}{2}$. This gives rise to,

$$\left(\sum_{3 \leq x_i \leq n} \frac{f((n+2)^2, x_i) - f(n^2, x_i)}{2x_i} \right) - dup + \pi(\mathcal{L}) + 1 = \frac{(n+2)^2 - n^2}{2}. \quad (4.2)$$

Equations (4.1) and (4.2) differ only by 1. These two equations will merge with a help of an ϵ variable that takes the value of 0 or 1 depending on the primality of $n+2$. But first, let's simplify equation (4.2) where $f((n+2)^2, x_i) = (n+2)^2 + 2x_i - ((n+2)^2 - x_i) \bmod(2x_i)$, and $f(n^2, x_i) = n^2 + 2x_i - (n^2 - x_i) \bmod(2x_i)$. The result after inserting these values into equation (4.2) is,

$$\sum_{3 \leq x_i \leq n} \frac{(n+2)^2 - n^2}{2x_i} + \sum_{3 \leq x_i \leq n} \frac{(n^2 - x_i) \bmod(2x_i) - ((n+2)^2 - x_i) \bmod(2x_i)}{2x_i} - \quad (4.3)$$

$$- dup + \pi(\mathcal{L}) = \frac{(n+2)^2 - n^2}{2} - 1.$$

For simplification, let $C_i = \frac{((n+2)^2 - x_i) \bmod(2x_i) - (n^2 - x_i) \bmod(2x_i)}{2x_i}$.

Hence, $\sum_{3 \leq x_i \leq n} \frac{(n^2 - x_i) \bmod(2x_i) - ((n+2)^2 - x_i) \bmod(2x_i)}{2x_i} = - \sum_{3 \leq x_i \leq n} C_i$.

Incorporate C_i into equation (4.3), and the result is,

$$\frac{(n+2)^2 - n^2}{2} \left(\sum_{3 \leq x_i \leq n} \frac{1}{x_i} - 1 \right) - \sum_{3 \leq x_i \leq n} C_i = dup - \pi(\mathcal{L}) - 1.$$

This means that,

$$\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = 2 \left(\frac{\text{dup} + \left(\sum_{3 \leq x_i \leq n} C_i \right) - \pi(\mathcal{L}) - 1}{(n+2)^2 - n^2} \right) + 1. \quad (4.4)$$

equation (4.4) comes from equation (4.3) which is a specific case when $n+2$ is a prime. Case 1 and 2 are merged into a general formula,

$$\boxed{\begin{aligned} \sum_{3 \leq x_i \leq n} \frac{1}{x_i} &= 2 \left(\frac{\text{dup} + \left(\sum_{3 \leq x_i \leq n} C_i \right) - \pi(\mathcal{L}) - \epsilon}{(n+2)^2 - n^2} \right) + 1. \quad (4.5) \\ \epsilon &= \begin{cases} 1, & \text{if } n+2 \text{ is prime} \\ 0, & \text{otherwise} \end{cases} \end{aligned}}$$

Equation (4.5) gives an exact formula that establishes a relation between the partial sum of the reciprocals of odd prime numbers $\sum_{3 \leq x_i \leq n} \frac{1}{x_i}$ and $\pi(\mathcal{L})$. This improved upon previous works of other authors, who established the upper and lower bounds of the partial sum of the reciprocals of prime numbers. Moreover, equation (4.5) opens a new window into the territories of the Legendre's Conjecture.

Example 4.1. What is the partial sum of the reciprocals of odd primes less than or equal to 11?

Solution: Let $n = 11$, so $(n+2) = 13$ is a prime. Using equation (4.5) where $n+2$ is prime means that $\epsilon = 1$. To have a better view, here is the list of odd composites between 11^2 and 13^2 .

11*11=121, (123 = 3, 41), (125 = 5, 5, 5), (129 = 3, 43), (133 = 7, 19), (135 = 3, 3, 3, 5), (141 = 3, 47), (143 = 11, 13), (145 = 5, 29), (147 = 3, 7, 7), (153 = 3, 3, 17), (155 = 5, 31), (159 = 3, 53), (161 = 7, 23), (165 = 3, 5, 11), **13*13=169**.

$$\text{dup} = \sum_{11^2 < n_k \leq 13^2} \text{dup}(n_k) = \text{dup}(123) + \text{dup}(125) + \dots + \text{dup}(169).$$

$$\text{dup}(n_k)_{11^2 < n_k \leq 13^2} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases}$$

where m is the number of distinct prime factors not exceeding $n = 11$, of the odd integer n_k .

Note: From the definition of $\text{dup}(n_k)$, we can see that, if n_k is a prime or a power of a prime, then $\text{dup}(n_k) = 0$. So we can skip n_k , if $n_k = p^\alpha$ where $\alpha \geq 1$ is an integer.

$\text{dup}(123 = 3 * 41) = 1 - 1 = 0$ because 123 has only one distinct prime factor less than or equal to 11. Thus, 123 has no duplicate count. It was only counted once as a multiple of 3. $\text{dup}(125 = 5 * 5 * 5) = 1 - 1 = 0$ because 125 has only one distinct prime factor that is less than or equal to 11, so 125 also was not counted more than once.

...

$\text{dup}(135 = 3 * 3 * 3 * 5) = 2 - 1 = 1$ because 135 has two distinct prime factors that are less than or equal to 11, these factors are 3 and 5. Hence, 135 was counted by $f(n, x)$ as a multiple of 3 and also as a multiple of 5. For these reasons, the extra count on 135 must be eliminated.

Similarly,

$$\text{dup}(147 = 3 * 7 * 7) = 2 - 1 = 1.$$

...

$$\text{dup}(165 = 3 * 5 * 11) = 3 - 1 = 2.$$

$\text{dup}(169 = 13 * 13) = 0$. the integer 169 has no prime factor less than or equal to $n = 11$.

- $\text{dup} = \sum_{11^2 < n_k \leq 13^2} \text{dup}(n_k) = 1 + 1 + 2 = 4$.
- $\pi(\mathcal{L}) = \pi((n+2)^2) - \pi(n^2) = 9$.
- $\sum_{3 \leq x_i \leq n} C_i = \sum_{3 \leq x_i \leq n} \frac{((n+2)^2 - x_i) \bmod (2x_i) - (n^2 - x_i) \bmod (2x_i)}{2x_i} = \frac{4-4}{2*3} + \frac{4-6}{2*5} + \frac{8-2}{2*7} + \frac{4-0}{2*11} = 0 - \frac{1}{5} + \frac{3}{7} + \frac{2}{11} = \mathbf{0.41039}$.
- $(n+2)^2 - n^2 = 13^2 - 11^2 = 169 - 121 = 48$.
- $\epsilon = 1$.
- $2 \left(\frac{\text{dup} + (\sum_{3 \leq x_i \leq n} C_i) - \pi(\mathcal{L}) - \epsilon}{(n+2)^2 - n^2} \right) + 1 = 2 \left(\frac{4 + (0.41039) - 9 - 1}{48} \right) + 1$.
- $2 \left(\frac{\text{dup} + (\sum_{3 \leq x_i \leq n} C_i) - \pi(\mathcal{L}) - \epsilon}{(n+2)^2 - n^2} \right) + 1 = \mathbf{0.7671}$. Therefore, by equation (6), $\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = \mathbf{0.7671}$.

Let's verify our result by calculating the value on the left side of equation (4.5).

$$\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = \sum_{3 \leq x_i \leq 11} \frac{1}{x_i} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} = \frac{886}{1155} = \mathbf{0.7671}.$$

Example 4.2. What is the partial sum of the reciprocals of odd primes less than or equal to 40?

Solution: Set $n = 37$ the largest prime less than or equal to 40, then $n + 2 = 39$. Using equation (4.5) where $n + 2$ is a composite gives $\epsilon = 0$.

Here is the list of the odd composite numbers between 37^2 and 39^2 .

37*37=1369, (1371 = 3, 457), (1375 = 5, 5, 5, 11), (1377 = 3, 3, 3, 3, 17), (1379 = 7, 197), (1383 = 3, 461), (1385 = 5, 277), (1387 = 19, 73), (1389 = 3, 463), (1391 = 13, 107), (1393 = 7, 199), (1395 = 3, 3, 5, 31), (1397 = 11, 127), (1401 = 3, 467), (1403 = 23, 61), (1405 = 5, 281), (1407 = 3, 7, 67), (1411 = 17, 83), (1413 = 3, 3, 157), (1415 = 5, 283), (1417 = 13, 109), (1419 = 3, 11, 43), (1421 = 7, 7, 29), (1425 = 3, 5, 5, 19), (1431 = 3, 3, 3, 53), (1435 = 5, 7, 41), (1437 = 3, 479), (1441 = 11, 131), (1443 = 3, 13, 37), (1445 = 5, 17, 17), (1449 = 3, 3, 7, 23), (1455 = 3, 5, 97), (1457 = 31, 47), (1461 = 3, 487), (1463 = 7, 11, 19), (1465 = 5, 293), (1467 = 3, 3, 163), (1469 = 13, 113), (1473 = 3, 491), (1475 = 5, 5, 59), (1477 = 7, 211), (1479 = 3, 17, 29), (1485 = 3, 3, 3, 5, 11), (1491 = 3, 7, 71), (1495 = 5, 13, 23), (1497 = 3, 499), (1501 = 19, 79), (1503 = 3, 3, 167), (1505 = 5, 7, 43), (1507 = 11, 137), (1509 = 3, 503), (1513 = 17, 89), (1515 = 3, 5, 101), (1517 = 37, 41), (1519 = 7, 7, 31), **39*39=1521**.

$$\text{dup} = \sum_{37^2 < n_k \leq 39^2} \text{dup}(n_k) = \text{dup}(1371) + \text{dup}(1375) + \dots + \text{dup}(1521)$$

$$\text{dup}(n_k)_{37^2 < n_k \leq 39^2} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases}$$

where m is the number of distinct prime factors not exceeding $n = 37$, of the odd integer n_k . In this example, only the $\text{dup}(n_k)$ that are not equal to zero are computed to avoid lengthy calculations.

$\text{dup}(1375 = 5 * 5 * 5 * 11) = 2 - 1 = 1$ because 1375 has two distinct prime factors that are less than or equal to 37.

$\text{dup}(1377 = 3 * 3 * 3 * 17) = 2 - 1 = 1$ because 1377 has two distinct prime factors that are less than or equal to 37.

$\text{dup}(1395 = 3 * 3 * 5 * 31) = 3 - 1 = 2$ because 1395 has three distinct prime factors that are less than or equal to 37.

$\text{dup}(1407 = 3 * 7 * 67) = 2 - 1 = 1$

$\text{dup}(1419 = 3 * 11 * 43) = 2 - 1 = 1$

$\text{dup}(1421 = 7 * 7 * 29) = 2 - 1 = 1$

$\text{dup}(1425 = 3 * 5 * 5 * 19) = 3 - 1 = 2$

$\text{dup}(1435 = 5 * 7 * 41) = 2 - 1 = 1$

$\text{dup}(1443 = 3 * 13 * 37) = 3 - 1 = 2$

$\text{dup}(1445 = 5 * 17 * 17) = 2 - 1 = 1$

$\text{dup}(1449 = 3 * 3 * 7 * 23) = 3 - 1 = 2$

$\text{dup}(1455 = 3 * 5 * 97) = 2 - 1 = 1$

$\text{dup}(1463 = 7 * 11 * 19) = 3 - 1 = 2$

$\text{dup}(1479 = 3 * 17 * 29) = 3 - 1 = 2$

$\text{dup}(1485 = 3 * 3 * 3 * 5 * 11) = 3 - 1 = 2$

$\text{dup}(1491 = 3 * 7 * 71) = 2 - 1 = 1$

$\text{dup}(1495 = 5 * 13 * 23) = 3 - 1 = 2$

$\text{dup}(1505 = 5 * 7 * 43) = 2 - 1 = 1$

$\text{dup}(1515 = 3 * 5 * 101) = 2 - 1 = 1$

$\text{dup}(1519 = 7 * 7 * 31) = 2 - 1 = 1$

$\text{dup}(1521 = 3 * 3 * 13 * 13) = 2 - 1 = 1$

• $\text{dup} = \sum_{37^2 < n_i \leq 39^2} \text{dup}(n_i) = \mathbf{29}$.

• $\pi(\mathcal{L}) = \pi((n+2)^2) - \pi(n^2) = \mathbf{21}$.

• $\sum_{3 \leq x_i \leq n} C_i = \sum_{3 \leq x_i \leq n} \frac{((n+2)^2 - x_i) \bmod(2x_i) - (n^2 - x_i) \bmod(2x_i)}{2x_i} = \frac{0-4}{2*3} + \frac{6-4}{2*5} + \frac{2-4}{2*7} + \frac{14-16}{2*11} + \frac{0-4}{2*13} + \frac{8-26}{2*17} + \frac{20-20}{2*19} + \frac{26-12}{2*23} + \frac{42-6}{2*29} + \frac{2-36}{2*31} + \frac{4-0}{2*37} = \mathbf{-0.952986}$.

• $(n+2)^2 - n^2 = 13^2 - 11^2 = 169 - 121 = \mathbf{152}$.

• $\epsilon = \mathbf{0}$.

$$2 \left(\frac{\text{dup} + \left(\sum_{3 \leq x_i \leq n} C_i \right) - \pi(\mathcal{L}) - \epsilon}{(n+2)^2 - n^2} \right) + 1 = 2 \left(\frac{29 + (-0.952986) - 21 - 0}{152} \right) + 1.$$

$$2 \left(\frac{\text{dup} + \left(\sum_{3 \leq x_i \leq n} C_i \right) - \pi(\mathcal{L}) - \epsilon}{(n+2)^2 - n^2} \right) + 1 = \mathbf{1.09272}.$$

Therefore, by equation (4.5), $\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = \boxed{1.09272}$. Let's verify our result by calculating the value on the left side of equation (4.5).

$$\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = \sum_{3 \leq x_i \leq 37} \frac{1}{x_i} = \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{37} = \frac{4054408822031}{3710369067405} = \boxed{1.09272}.$$

More Examples without detailed calculations.

Example: Compute the partial sum of reciprocals of odd primes not exceeding $n = 41$.

- $\sum_{3 \leq x_i \leq n} C_i = -0.162415$.
- $\text{dup} = 31$.
- $\pi(\mathcal{L}) = 20$.
- $(43^2 - 41^2) = 168$.
- $\epsilon = 1$.
- Sum from the right side of equation 4.5 is, **1.11711**.
- $\sum_{3 \leq x_i \leq 41} \frac{1}{x_i} = \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{41} = \mathbf{1.11711}$.

Example: Compute the partial sum of reciprocals of odd primes not exceeding $n = 43$.

- $\sum_{3 \leq x_i \leq n} C_i = -1.64745$.
- $\text{dup} = 37$.
- $\pi(\mathcal{L}) = 23$.
- $(45^2 - 43^2) = 176$.
- $\epsilon = 0$.
- Sum from the right side of equation 4.5 is, **1.14037**.
- $\sum_{3 \leq x_i \leq 43} \frac{1}{x_i} = \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{43} = \mathbf{1.14037}$.

Example: Compute the partial sum of reciprocals of odd primes not exceeding $n = 45$.

- $\sum_{3 \leq x_i \leq n} C_i = 1.91403$.
- $\text{dup} = 35$.
- $\pi(\mathcal{L}) = 23$.
- $(47^2 - 45^2) = 184$.
- $\epsilon = 1$.
- Sum from the right side of equation 4.5 is, **1.14037**.
- $\sum_{3 \leq x_i \leq 45} \frac{1}{x_i} = \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{43} = \mathbf{1.14037}$.

4.2. Counting the odd integers in the interval $(n^2, n(n+2)]$. This count give an exact formula of $\pi(\mathcal{L})_1 = \pi(n(n+2)) - \pi(n^2) = \pi(n+1)^2 - \pi(n^2)$, the number of primes between two consecutive squares when starting from an odd square n^2 .

Note: The number of primes in the interval $(n^2, n(n+2)]$ is the same as the number of prime in the interval $(n^2, (n+1)^2]$ since $n(n+2)+1 = (n+1)^2$ and $(n+1)^2$ is an even number. We use $n(n+2)$ because $f(n, x)$ takes only odd integers.

The new equation is built by counting the odd integers in the interval $(n^2, n(n+2)]$. To avoid repeating the process of equation (4.5), we simply deduce this new equation from (4.5). We replace the upper bound $(n+2)^2$ in equation (4.5) with the new upper bound $n(n+2)$. The variables in the new equation have similar definitions to those in equation (4.5), but the interval here is $(n^2, n(n+2)]$ instead of $(n^2, (n+2)^2]$.

$$\text{dup}(n_k)_{n^2 < n_k \leq n(n+2)} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases}$$

where m is the number of distinct prime factors not exceeding n of the odd integer n_k .

$$\begin{aligned} (dup)_1 &:= \sum_{n^2 < n_k \leq n(n+2)} dup(n_k). \\ \pi(\mathcal{L})_1 &= \pi(n(n+2)) - \pi(n^2). \\ C_1 &= \frac{(n(n+2) - x_i) \bmod(2x_i) - (n^2 - x_i) \bmod(2x_i)}{2x_i}. \end{aligned}$$

$$\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = 2 \left(\frac{(dup)_1 + \left(\sum_{3 \leq x_i \leq n} C_1 \right) - \pi(\mathcal{L})_1}{n(n+2) - n^2} \right) + 1. \quad (4.6)$$

4.3. Counting the odd integers in the interval $(n(n+2), (n+2)^2]$. This count gives an exact formula of $\pi(\mathcal{L})_2 = \pi((n+2)^2) - \pi(n(n+2)) = \pi((n+2)^2) - \pi((n+1)^2)$, the number of primes between two consecutive squares when starting from an even square $(n+1)^2$. The equation obtained by counting the odd integers in the interval $(n(n+2), (n+2)^2]$ is deduced from (4.5) by replacing n^2 with $n(n+2)$. The variables in the new equation have similar definitions to those in (4.5).

$$dup(n_k)_{n(n+2) < n_k \leq (n+2)^2} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases} :$$

where m is the number of distinct prime factors not exceeding n of the odd integer n_k .

$$\begin{aligned} (dup)_2 &:= \sum_{n(n+2) < n_k \leq (n+2)^2} dup(n_k). \\ \pi(\mathcal{L})_2 &= \pi((n+2)^2) - \pi(n(n+2)). \\ C_2 &= \frac{((n+2)^2 - x_i) \bmod(2x_i) - (n(n+2) - x_i) \bmod(2x_i)}{2x_i}. \end{aligned}$$

$$\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = 2 \left(\frac{(dup)_2 + \left(\sum_{3 \leq x_i \leq n} C_2 \right) - \pi(\mathcal{L})_2 - \epsilon}{(n+2)^2 - n(n+2)} \right) + 1. \quad (4.7)$$

$$\epsilon = \begin{cases} 1, & \text{if } n+2 \text{ is prime} \\ 0, & \text{otherwise} \end{cases}$$

4.4. A possible path to the Legendre's Conjecture. We obtain an exact formula of the partial sum of the reciprocals of odd primes in equation (4.6). This is in terms of $\pi(\mathcal{L})_1$ the number of primes between n^2 and $(n+1)^2$. One can argue that,

- If $\pi(\mathcal{L})_1 = 0$, then $\sum_{3 \leq x_i \leq n} \frac{1}{x_i}$ would be bigger than its own upper bound. And that is impossible. Hence, there must be a prime between n^2 and $(n+1)^2$. But one would need a good upper bound and a descent approximation of $(dup)_1$. A similar reasonment can be made with $\pi(\mathcal{L})_2$ in equation (4.7).

5. REFINEMENT OF THE PRIME COUNTING FUNCTION

This is another application of $f(n, x)$ as a counting function. A refinement of the prime counting function leads to a better understanding of the distribution of prime numbers. It also increases the scope to access and wrestle down some open problems in prime numbers. The correctness and precision of the results in examples 10 and 11, echo on the strength of equation (4.5). We shall build a similar equation by counting the odd integers in the interval $(1, n^2]$ instead of $(n^2, (n+2)^2]$.

Let x_n denote the largest primes not exceeding the ceiling of the square root of n . The number of odd composites in the interval $(1, n^2]$ is the number of odd composite multiples of 3, plus the number of odd composite multiples of 5 ..., plus the number of odd composite multiples of x_n , and we subtract the duplicates. The odd composite multiples of 3 not exceeding n^2 start from 3 where 3 is not included, and extend to the largest odd multiple of 3 not exceeding n^2 which is given by $n^2 - (n^2 - 3) \bmod(2 * 3)$. Thus, the number of odd composite multiples to 3 not exceeding n^2 is,

$$\frac{n^2 - (n^2 - 3) \bmod(2 * 3) - 3}{2 * 3}.$$

Similarly, the number of odd composite multiples of 5 less than or equal to n^2 is,

$$\frac{n^2 - (n^2 - 5) \bmod(2 * 5) - 5}{2 * 5}.$$

In general, given a prime x_i such that $3 \leq x_i \leq n$ the number of odd composite multiples of x_i not exceeding n^2 is,

$$\frac{n^2 - (n^2 - x_i) \bmod(2x_i) - x_i}{2x_i}.$$

Consequently, the number of odd composites greater than 1 and not exceeding n^2 is,

$$\left(\sum_{3 \leq x_i \leq n} \frac{n^2 - (n^2 - x_i) \bmod(2x_i) - x_i}{2x_i} \right) - dup.$$

$$dup = \sum_{1 < n_k \leq n^2} dup(n_k).$$

$$dup(n_k)_{1 < n_k \leq n^2} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases}$$

where m is the number of distinct prime factors less than or equal to n , of the odd integer n_k .

The total number of odd integers in the interval $(1, n^2]$ is,

$$\begin{aligned} & \left(\sum_{3 \leq x_i \leq n} \frac{n^2 - (n^2 - x_i) \bmod(2x_i) - x_i}{2x_i} \right) - dup + \pi(n^2). \\ & \sum_{3 \leq x_i \leq n} \frac{n^2}{2x_i} - \left(\sum_{3 \leq x_i \leq n} \frac{(n^2 - x_i) \bmod(2x_i) + x_i}{2x_i} \right) - dup + \pi(n^2). \end{aligned}$$

$$\text{Set } B_i = \frac{(n^2 - x_i) \bmod(2x_i) + x_i}{2x_i}.$$

$$\sum_{3 \leq x_i \leq n} \frac{n^2}{2x_i} - \sum_{3 \leq x_i \leq n} B_i - \text{dup} + \pi(n^2).$$

The total number of odd integers in the interval $(1, n^2]$ is also,

$$\frac{n^2 - 1}{2}.$$

We obtain the equality,

$$\sum_{3 \leq x_i \leq n} \frac{n^2}{2x_i} - \sum_{3 \leq x_i \leq n} B_i - \text{dup} + \pi(n^2) = \frac{n^2 - 1}{2}.$$

This simplifies to,

$$\sum_{3 \leq x_i \leq n} \frac{1}{x_i} = 2 \left(\frac{\text{dup} + \left(\sum_{3 \leq x_i \leq n} B_i \right) - \pi(n^2) - \frac{1}{2}}{n^2} \right) + 1.$$

$$\boxed{\pi(n^2) = \text{dup} + \left(\sum_{3 \leq x_i \leq n} B_i \right) - \frac{1}{2} - \frac{n^2}{2} \left(\sum_{3 \leq x_i \leq n} \frac{1}{x_i} - 1 \right)}. \quad (5.1)$$

Note: The prime 2 is not included in the count as we only worked with odd integers. Equation (5.1) is a refinement of the prime counting function it is an "exact formula", and it gives more detailed information. We can see that $\pi(n^2)$ is expressed in terms of some important values with historic interests such as the partial sum of the reciprocals of odd primes inferior or equal to n , and the dup function which is derived from a well known function $\omega(n)$.

Example 5.1. Compute $\pi(7^2)$.

$$\text{dup} = \sum_{1 < n_k \leq 49} \text{dup}(n_k) = \text{dup}(3) + \text{dup}(5) + \text{dup}(7) + \text{dup}(9) + \dots + \text{dup}(49).$$

By the definition of the dup function, we know that if n_k is a prime or a power of a prime, then $\text{dup}(n_k) = 0$. So we can skip n_k , if $n_k = p^\alpha$ where $\alpha \geq 1$ is an integer.

$$\text{dup}(15) = \text{dup}(3 * 5) = 2 - 1 = 1.$$

$$\text{dup}(21) = \text{dup}(3 * 7) = 2 - 1 = 1.$$

$$\text{dup}(33) = \text{dup}(3 * 11) = 1 - 1 = 0.$$

$$\text{dup}(35) = \text{dup}(5 * 7) = 2 - 1 = 1.$$

$$\text{dup}(39) = \text{dup}(3 * 13) = 1 - 1 = 0.$$

$$\text{dup}(45) = \text{dup}(3 * 3 * 5) = 2 - 1 = 1.$$

$$\bullet \text{dup} = \sum_{1 < n_k \leq 49} \text{dup}(n_k) = 1 + 1 + 1 + 1 = 4.$$

$$\bullet \sum_{3 \leq x_i \leq n} B_i = \frac{(7^2 - 3) \bmod(2 * 3) + 3}{2 * 3} + \frac{(7^2 - 5) \bmod(2 * 5) + 5}{2 * 5} + \frac{(7^2 - 7) \bmod(2 * 7) + 7}{2 * 7} = \frac{7}{6} + \frac{9}{10} + \frac{1}{2}.$$

$$\bullet \sum_{3 \leq x_i \leq 7} \frac{1}{x_i} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7}.$$

$$\begin{aligned} \pi(n^2) &= \text{dup} + \left(\sum_{3 \leq x_i \leq n} B_i \right) - \frac{1}{2} - \frac{n^2}{2} \left(\sum_{3 \leq x_i \leq n} \frac{1}{x_i} - 1 \right). \\ &= 4 + \left(\frac{7}{6} + \frac{9}{10} + \frac{1}{2} \right) - \frac{1}{2} - \frac{7^2}{2} \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} - 1 \right). \\ \pi(7^2) &= 14. \end{aligned}$$

5.1. Comparing $\pi(x)$ with its refinement.

The prime number Theorem, first proved by Hadamard [6] and de la Vallee-Poussin [9] states that

$$\pi(x) \sim \frac{x}{\ln x} \quad \text{as } x \rightarrow \infty.$$

And in its strongest known form, the prime number Theorem is a statement that

$$\pi(x) = \text{Li}(x) + R(x). \quad (5.2)$$

where

$$\text{Li}(x) := \int_2^x \frac{dt}{\log t} = x \left(\frac{1}{\log x} + \frac{1!}{\log^2 x} + \dots + \frac{m!}{\log^{m+1} x} + \mathcal{O} \left(\frac{1}{\log^{m+2} x} \right) \right)$$

for any fixed integer $m \geq 0$ and

$$R(x) \ll x \exp(-C\delta(x)), \quad \delta(x) := (\log x)^{3/5} (\log \log x)^{-1/5} (C > 0) \text{ [8]}.$$

The function $f(n, x_i)$ where n is a positive odd integer greater than 3 and x_i is an odd prime less than or equal to the ceiling of the square root of n , states that,

$$\pi(n^2) = \text{dup} + \left(\sum_{3 \leq x_i \leq n} B_i \right) - \frac{1}{2} - \frac{n^2}{2} \left(\sum_{3 \leq x_i \leq n} \frac{1}{x_i} - 1 \right). \quad (5.3)$$

where

$$\text{dup} = \sum_{1 < n_k \leq n^2} \text{dup}(n_k).$$

$$\text{dup}(n_k)_{1 < n_k \leq n^2} = \begin{cases} 0, & \text{if } m = 0 \\ m - 1, & \text{otherwise} \end{cases}$$

where m is the number of distinct prime factors less than or equal to n , of the odd integer n_k .

$$B_i = \frac{(n^2 - x_i) \bmod (2x_i) + x_i}{2x_i}.$$

- Equation (5.2) from the prime number Theorem gives an approximation of $\pi(x)$, while equation (5.3) is an "exact formula" of $\pi(n^2)$.

- Equation (5.3) can be useful to introduce the prime counting function without going much deeper into the Riemann Hypothesis.

- The proof of equation (5.3) is elementary, it follows the counting process we used to obtain the equation.

- Equation (5.3) yields much deeper insights into the distribution of the prime numbers. But to have an exact value of $\pi(n^2)$, one may have to count by hand the dup function. This may be adequate for small values of n as illustrated in examples 10 and 11. But large values of n would challenge us to develop and improve an approximation of the dup function.

- "The precise behavior of $\pi(x) - \text{Li}(x)$ in equation (5.2) depends on the location of the zeros of the Riemann zeta function, and cannot be determined until we have more precise information about them than we have now." [2] While the precision of $\pi(n^2)$ in equation (5.3)

(for large values of n) depends on the approximation of $\text{dup} = \sum_{1 < k \leq n^2} \text{dup}(k)$ that is related to the summatory function of $\omega(k)$ given by $\sum_2^n \omega(k) = n \ln \ln n + B_1 n + \mathcal{O}\left(\frac{n}{\ln n}\right)$. Where B_1 is a Mertens constant [10].

- Equation (5.3) has its limitations. Computing $\pi(n^2)$ requires that we know the primes not exceeding n to calculate $\sum_{3 \leq p_i \leq n} \frac{1}{p_i}$ and $\sum_{3 \leq p_i \leq n} B_i$. But the presence of $\sum_{3 \leq p_i \leq n} \frac{1}{p_i}$ in the formula of $\pi(n^2)$ can also be an asset if it is used as a stepping stone to the Riemann's Hypothesis. Moreover, equation (5.3) is in terms of $\text{dup} = \sum_{1 < k \leq n^2} \text{dup}(k)$ that we don't know its exact formula. But due to Hardy, Ramanujan [7] and Erdos [5], we have good information on $\omega(n)$ that can be useful to approximate the dup function.

The number of distinct prime factors of a given integer n , is to equation (5.3) what the location of the zeros in the Riemann zeta function is to the prime number Theorem. It opens new windows into more accurate and precise estimations of the prime counting function and refines our understanding of the distribution of prime numbers. According to some experts, the difficulty of the Riemann's Hypothesis is as follows:

"There is no approach currently known to understand the distribution of prime numbers well enough to establish the desired approximation, other than by studying the Riemann zeta function and its zeros." [4]. Equation (5.3) can be an alternative approach, and it can bring new perspectives in future investigations of the Riemann's Hypothesis.

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